Analysis 1, Summer 2023

List 7

L'Hôpital's Rule, Taylor series and polynomials

168. Give an equation for the tangent line to $y = e^{3x}(\cos(4x))^5$ at $x = 1$. $\boxed{y = 3x + 1}$

169. Is $y = e^{\sin(x)}$ concave up or concave down when $x = \pi$? $\boxed{\text{concave down}}$ because $f''(\pi) = -1$ is negative.

170. Find the absolute extremes of $x \ln(x)$ on...

    (a) the interval $[0, \frac{1}{2}]$. $\boxed{\text{abs min at } x = \frac{1}{e}\text{, abs max at } x = 0}$

    (b) the interval $[0, 1]$. $\boxed{\text{abs min at } x = \frac{1}{e}\text{, abs max at both } x = 0, x = 1}$

    (c) the interval $[0, 2]$. $\boxed{\text{abs min at } x = \frac{1}{e}\text{, abs max at } x = 2}$

    (d) the interval $[1, 2]$. $\boxed{\text{abs min at } x = 1\text{, abs max at } x = 2}$

171. Find the inflection points of $f(x) = \frac{3}{10}x^5 - 5x^4 + 32x^3 - 96x^2 + 28$. $\boxed{x = 2}$ is the only inflection point. Although $f''(4) = 0$, the sign of f'' does not *change* at $x = 4$ because $f''(x) > 0$ for all x near 4.

172. If f is a smooth function with

x	-2	-1	0	1	2	3	4
f	3	5	-3	7	8	9	12
f'	2	0	-1	-1	1	3	0
f''	0	4	1	-1	$\frac{-8}{3}$	0	1

answer the following:

    (a) Does f have a critical point at $x = 0$? $\boxed{\text{No}}$ $f'(0) \neq 0$

    (b) Does f have a local minimum at $x = -1$? $\boxed{\text{Yes}}$ $f'(-1) = 0$ and $f''(-1) > 0$

    (c) Does f have a local maximum at $x = 4$? $\boxed{\text{No}}$ local min

    (d) It it possible that f has an absolute minimum at $x = -1$? $\boxed{\text{No}}$ $f(x) < f(-1)$ for $x = -2$ and $x = 0$

    (e) It it possible that f has an absolute maximum at $x = -1$? $\boxed{\text{No}}$ $f(x) > f(-1)$ for $x = 1, 2, \ldots$

    (f) It it possible that f has an inflection point at $x = 3$? $\boxed{\text{Yes}}$ $f''(3) = 0$ and $f''(2) < 0 < f''(4)$ (f'' changes sign)

    (g) It it possible that f has an inflection point at $x = 4$? $\boxed{\text{No}}$ $f''(4) \neq 0$

L'Hôpital's Rule: if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The same substitution works if $\lim_{x\to a} f(x) = \infty$ or $-\infty$ and $\lim_{x\to a} g(x) = \infty$ or $-\infty$. And also for one-sided limits and for $x \to \infty$ and $x \to -\infty$.

173. Calculate $\lim_{x\to 1} \frac{3x^3+4x^2-13x+6}{2x^4+x^3-x^2+x-3}$ and $\lim_{x\to 4} \frac{\sin(\pi x)}{\ln(x-3)}$.

$\lim_{x\to 1} \frac{9x^2+8x-13}{8x^3+3x^2-2x+1} = \frac{9+8-13}{8+3-2+1} = \frac{4}{10} = \boxed{\frac{2}{5}}$

$\lim_{x\to 4} \frac{\pi\cos(\pi x)}{1/(x-3)} = \frac{\pi\cos(4\pi)}{1/1} = \boxed{\pi}$

174. Calculate the following limits:

(a) $\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$ L'Hôpital: $\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$.
Then $\frac{1/x}{-1/x^2} = \frac{-x^2}{x} = -x$, so the limit is $-0 = \boxed{0}$.

(b) $\lim_{x\to 0^+} x\ln(x)$ Algebra: $x\ln(x) = \frac{\ln(x)}{1/x}$, so this is the same as (a). $\boxed{0}$

(c) $\lim_{x\to 0^+} e^{x\ln(x)} = e^{\left(\lim_{x\to 0^+} x\ln(x)\right)} = e^0 = \boxed{1}$

(d) $\lim_{x\to 0^+} x^x$ Algebra: $x^x = (e^{\ln(x)})^x = e^{x\ln(x)}$, so this is the same as (c). $\boxed{1}$

Hint for (c): recall that $\lim_{x\to a} f(g(x)) = f\big(\lim_{x\to a} g(x)\big)$ if f is continuous.

175. (a) Find $\lim_{x\to 1} \frac{x^2-18}{3x+4}$. $\frac{1-18}{3+4} = \boxed{\frac{-17}{7}}$

Find $\lim_{x\to 1} \frac{2x}{3}$. $\frac{2(1)}{3} = \boxed{\frac{2}{3}}$

(b) Why are the answers to (a) and (b) not equal? Because $\frac{x^2-18}{3x+4}$ is NOT $\frac{0}{0}$ when $x=1$.

176. Find $\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)} &= \lim_{x\to 0} \frac{2\cos(x)-2\cos(2x)}{1-\cos(x)} \\
&= \lim_{x\to 0} \frac{-2\sin(x)+4\sin(2x)}{\sin(x)} \\
&= \lim_{x\to 0} \frac{-2\cos(x)+8\cos(2x)}{\cos(x)} \\
&= \frac{-2+8}{1} = \boxed{6}
\end{aligned}$$

177. (a) Calculate $\lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n}\right)$ using L'Hôpital. $\boxed{1}$

(b) Calculate $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ using the fact that $f(n) = e^{\ln(f(n))}$ and therefore

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} e^{\ln(f(n))} = e^{\left(\lim\limits_{n\to\infty} \ln(f(n))\right)}.$$

$e^1 = \boxed{e}$

178. For the function $f(x) = x^2 e^{-x}$, find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

(a) $\lim\limits_{x\to\infty} x^2 e^{-x} = \lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \overset{\text{L'H}}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \overset{\text{L'H}}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = \boxed{0}$.

(b) Do not use L'Hôpital! Just plug in $x = 0$ to get $\frac{0^2}{e^{-0}} = \frac{0}{e^0} = \frac{0}{1} = \boxed{0}$.

(c) $\lim\limits_{x\to-\infty} x^2 e^{-x} = \lim\limits_{x\to\infty} x^2 e^x = \boxed{+\infty}$.

For a function $f(x)$, the **degree-N Taylor polynomial around $x = a$** is

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $n! = n \cdot (n-1) \cdots 2 \cdot 1$ is a factorial and $f^{(n)}$ is the n^{th} derivative of f. Note that $0! = 1$ and that $f^{(0)} = f$. In expanded form, this is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

179. (a) Calculate the functions $f'(x)$ and $f''(x)$ for $f(x) = x^{5/2}$.

(b) Calculate the numbers $f(4)$, $f'(4)$, and $f''(4)$ for $f(x) = x^{5/2}$.

(c) Give the degree-2 Taylor polynomial for $x^{5/2}$ around $x = 4$. (You may leave "$(x-4)$" in your answer; you do not have to expand it to "__x^2 + ...".)

$\boxed{32 + 20(x-4) + \dfrac{15}{4}(x-4)^2}$

180. Give the degree-3 Taylor polynomial for $e^x \cos(x)$ around $x = 0$. $\boxed{1 + x - \dfrac{x^3}{3}}$

(You will first need to find $f'(x)$, $f''(x)$, $f'''(x)$ and the numbers $f(0)$, ..., $f'''(0)$.)

181. (a) Give the quadratic Taylor polynomial for $\sqrt{x}$ around $x = 1$.

$\boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2}$

(b) Plug $x = 1.2$ into your polynomial from part (a) to get a "quadratic approximation" to $\sqrt{1.2}$. $1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 = \boxed{1.095}$

(c) Compare the quadratic approximation to the linear approximation from Task 82(a)-(c). Which is closer to the true value of $\sqrt{1.2} \approx 1.09545$? The quadratic approximation is better (closer to the exact value $\sqrt{1.2}$) than the linear approximation.

The **Taylor series around $x = a$** is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. Here are Taylor[1] series around zero for some common functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

182. Give the Taylor series for $\frac{x^3}{1-x}$ around $x = 0$. $(x^3)(\frac{1}{1-x}) = x^3(1 + x^2 + x^3 + x^4 + x^5 + \cdots) = \boxed{x^3 + x^5 + x^6 + x^7 + x^8 + \cdots}$

183. Give the Taylor series for $\ln(1+x^2)$ around $x = 0$. $\boxed{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \cdots}$

184. Give the Taylor polynomial of degree 6 for $f(x) = \ln(x)$ around $x = 1$.
$\boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6}$

185. (a) Give the Taylor polynomial of degree 3 for $f(x) = \frac{x}{\cos(x)}$ around $x = 0$.

$$f(x) = \frac{x}{\cos(x)} \qquad f(0) = 0$$

$$f'(x) = \frac{\cos(x) + x\sin(x)}{(\cos(x))^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{\mathrm{c}^2(-\mathrm{s} + x\mathrm{c} + \mathrm{s}) - (\mathrm{c} + x\mathrm{s})2\mathrm{c}(-\mathrm{s})}{\mathrm{c}^4} \qquad f''(0) = 0$$

$$= \frac{x\mathrm{c}^3 + 2\mathrm{c}^2\mathrm{s} + 2x\mathrm{c}\mathrm{s}^2}{\mathrm{c}^4}$$

$$f'''(x) = \ldots \qquad f'''(0) = 3$$

so $P(x) = 0 + 1x + \frac{0}{2}x^2 + \frac{3}{3!}x^3 = \boxed{x + \frac{1}{2}x^3}$

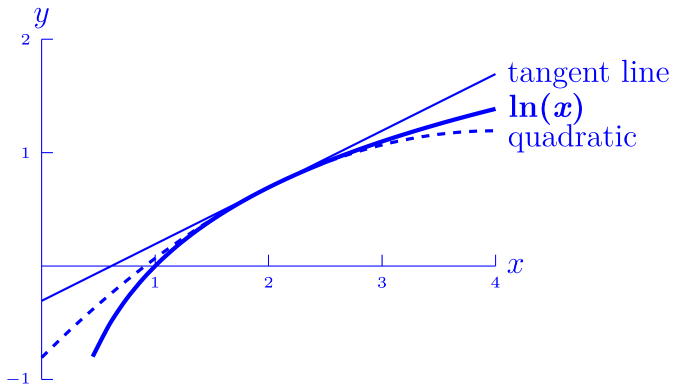
(b) Give the Taylor polynomial of degree 4 for $f(x) = \frac{\sin(x)}{x}$ around $x = 0$.

$\frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{7!}x^6 + \cdots$, so the Taylor polynomial is $\boxed{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4}$

(c) Which more difficult—part (a) or part (b)? $\boxed{\text{(a)}}$

186. On a single set of axes with $x \in [0, 4]$ and $y \in [-1, 2]$, draw the curve $y = \ln(x)$, the tangent line to $y = \ln(x)$ at the point $(2, \ln 2)$, and the graph of the quadratic Taylor polynomial for $\ln(x)$ around $x = 2$.

[1] A Taylor series around zero is also called a "Maclaurin series".



An **anti-derivative** of $f(x)$ is a function whose derivative is $f(x)$.
In symbols, $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

187. (a) Give an anti-derivative of $10x^9$.
That is, give a function $F(x)$ for which $F'(x) = 10x^9$.

(b) Give another anti-derivative of $10x^9$.

(c) Give another anti-derivative of $10x^9$.

(d) Give another anti-derivative of $10x^9$.

All answers will be of the form $\boxed{x^{10} + C}$. These might include x^{10} or $x^{10} + 1$ or $x^{10} - 12345$, etc.

188. Give an anti-derivative of $\sin(x)$. $\boxed{-\cos(x)}$ or any $-\cos(x) + C$

189. Give an anti-derivative for each of the following functions:

(a) x^3 $\boxed{\frac{1}{4}x^4}$

(b) $12x^5$ $\boxed{2x^6}$

(c) $12x^4$ $\boxed{\frac{12}{5}x^5}$

(d) x^{15} $\boxed{\frac{1}{16}x^{16}}$

(e) $-3x^{15}$ $\boxed{\frac{-3}{16}x^{16}}$

(f) $\frac{1}{2}x^2$ $\boxed{\frac{1}{6}x^3}$

(g) x^{5000} $\boxed{\frac{1}{5001}x^{5001}}$

(h) $\frac{3}{5}x^{12}$ $\boxed{\frac{3}{65}x^{13}}$

(i) $\frac{-4}{3}x^7$ $\boxed{\frac{-1}{6}x^8}$

(j) $5\sin(x)$ $\boxed{-5\cos(x)}$

(k) $2\cos(x)$ $\boxed{2\sin(x)}$

(ℓ) e^x $\boxed{e^x}$

190. Give an anti-derivative of $3x^2\cos(x^3 + 9)$. $\boxed{\sin(x^3 + 9)}$